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Ladder operators and differential equations for orthogonal polynomials

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Abstract. Under some integrability conditions we derive raising and lowering differential recurrence relations for polynomials orthogonal with respect to a weight function supported in the real line. We also derive a second-order differential equation satisfied by these polynomials. We discuss the Lie algebra generated by the generalized creation and annihilation operators. From the differential equations, Plancherel–Rotach type asymptotics are derived. Under certain conditions, stated in the text, an Airy function emerges.

1. Introduction

We first mention a physical motivation for considering the mathematical questions addressed in this paper. Let $\{p_n(x)\}_{n\geqslant 0}$ be a set of polynomials orthonormal with respect to the weight function $w(x)$ defined for $x \in \mathbb{R}$

$$\int_{-\infty}^{\infty} p_m(x)p_n(x)w(x)\,\mathrm{d}x = \delta_{mn}. \tag{1.1}$$

We also write

$$w(x) = \mathrm{e}^{-v(x)}. \tag{1.2}$$

In the theory of $N \times N$ Hermitean matrix models, which has application in quantum gravity, transport in disordered systems and quantum chaos, a fundamental quantity, $E[J]$, which gives the probability that an interval J (a subset of $\mathbb{R}$) is free of eigenvalues is of great interest. This quantity can be expressed as the Fredholm determinant of a certain integral operator over the interval J:

$$E[J] := \det(I - K_J)$$

where K has kernel,

$$K_N(x, y) = \sqrt{w(x)w(y)}\,\frac{p_N(x)p_{N-1}(y) - p_N(y)p_{N-1}(x)}{x - y}$$

that is

$$K_N(x, y) = \sqrt{w(x)w(y)}\,\frac{\gamma_N}{\gamma_{N-1}}\sum_{k=0}^{N-1} p_k(x)p_k(y)$$

where γ_n is the coefficient of x^n in $p_n(x)$.

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In the case where $J = (s, \infty)$ and s is an appropriately scaled (or dimensionless) variable, $E[J]$, is related to the probability of finding the largest eigenvalue at s. For the cases of the Hermite ($w(x) = \mathrm{e}^{-x^2}$) and the Laguerre weight ($w(x) = x^\alpha\,\mathrm{e}^{-x}$) functions it was shown by Tracy and Widom [23] that the scaled kernel becomes

$$K(x, y) = \frac{A(x)A'(y) - A(y)A'(x)}{x - y}$$

where $A(x)$ is the Airy function. The Airy function arises because the Plancherel–Rotach (or the uniform) asymptotics of the Hermite and Laguerre polynomials involve Airy functions. For the case of the 'Airy kernel' the logarithm of $E[s, \infty]$ satisfies a particular Painlevé II transcendent [23]. It is therefore of interest to determine if the uniform asymptotics for polynomials orthonormal with respect to

$$w(x) := \mathrm{e}^{-v(x)} \qquad v(x) = x^{2m} + \text{a polynomial of lower degree}$$

gives rise to an Airy function. In this paper we give an affirmative answer to the above question. By using a differential recurrence relation, to be shown later, from which a second-order ordinary differential equation satisfied by the polynomials can be derived. We will show that by transforming the ordinary differential equation into the Schrödinger form, the one-particle potential has a linear turning point, valid for sufficiently large degree of the polynomials and uniformly in x.

In this paper we assume $v(x)$ of (1.2) to be twice continuously differentiable and convex for $x \in \mathbb{R}$. As a consequence of the orthogonality, $\{p_n(x) : n \geqslant 0\}$ satisfies the three-term recurrence relation,

$$x p_n(x) = \sqrt{\beta_{n+1}}\, p_{n+1}(x) + \alpha_n p_n(x) + \sqrt{\beta_n}\, p_{n-1}(x) \qquad x \in \mathbb{R}. \tag{1.3}$$

In the notation of Nevai [17],

$$a_n = \sqrt{\beta_n}. \tag{1.4}$$

We shall use both notations throughout this work. In section 2 we derive the differential recurrence relation

$$p_n'(x) = -B_n(x) p_n(x) + A_n(x) p_{n-1}(x) \tag{1.5}$$

where

$$A_n(x) := a_n \int_{-\infty}^{\infty} \frac{v'(x) - v'(y)}{x - y}\, p_n^2(y) w(y)\,\mathrm{d}y \tag{1.6}$$

and

$$B_n(x) := a_n \int_{-\infty}^{\infty} \frac{v'(x) - v'(y)}{x - y}\, p_n(y) p_{n-1}(y) w(y)\,\mathrm{d}y. \tag{1.7}$$

If $v(x)$ is a polynomial it is clear from (1.3), (1.6) and (1.7) that $A_n(x)$ and $B_n(x)$ are polynomials. The differentiation formula, (1.5), is trivially,

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} + B_n(x)\right) p_n(x) = A_n(x) p_{n-1}(x). \tag{1.8}$$

Thus

$$L_{n,1} := \left(\frac{\mathrm{d}}{\mathrm{d}x} + B_n(x)\right) \tag{1.9}$$

is an annihilation operator. A creation operator can be found using the three-term recurrence relation (1.3). In section 2 we shall prove that

$$\left(-\frac{\mathrm{d}}{\mathrm{d}x} + B_n(x) + v'(x)\right) p_{n-1}(x) = A_{n-1}(x)\sqrt{\frac{\beta_n}{\beta_{n-1}}}\, p_n(x) = A_{n-1}(x)\frac{a_n}{a_{n-1}} p_n(x). \tag{1.10}$$

Therefore the creation operator is

$$L_{n,2} := \left(-\frac{\mathrm{d}}{\mathrm{d}x} + B_n(x) + v'(x)\right). \tag{1.11}$$

It is important to note that the creation and annihilation operators are adjoint when the underlying Hilbert space has the inner product

$$\langle f, g\rangle := \int_{-\infty}^{\infty} f(y)\overline{g(y)}w(y)\,\mathrm{d}y. \tag{1.12}$$

The creation and annihilation operators for the classical orthogonal polynomials are also adjoint and it is interesting that this property continues to hold in our general setting.

In section 2 we combine (1.8) and (1.10) to derive a second-order differential equation satisfied by $\{p_n(x)\}$. The differential equation is stated as (2.6). Atkinson and Everitt [2] revisited a method of Shohat [20] to derive a linear second-order differential equation with polynomial coefficients satisfied by the orthogonal polynomials when $v'(x)$ is a rational function. In section 2 we show that, when $v'(x)$ is a rational function, our differential recurrence relations (1.8) and (1.10) will lead to a linear second-order differential equation with polynomial coefficients, thus giving an alternative approach to Shohat's construction [20]. The differential equation is stated in theorem 2.2. Our construction requires the knowledge of the recurrence coefficients $\{a_n\}$ while Shohat's requires the knowledge of the Stieltjes transform of w and the numerator polynomials. Our approach is particularly advantageous in studying qualitative properties of the orthogonal polynomials when the large n asymptotics of a_n are known.

In section 3 we study the Lie algebra generated by the raising and lowering operators $L_{n,1}, L_{n,2}$. This Lie algebra has dimension $2m+1$ when $v(x)$ is a polynomial of degree $2m$. We determine the conditions under which this Lie algebra is finite dimensional.

It is important to note that we have essentially derived a Rodrigues formula for general orthogonal polynomials. This is the case since (1.10) implies

$$R_{n,2}R_{n-1,2}\ldots R_{1,2}[\exp(-v(x))] = (-1)^n\left[\prod_{k=1}^{n} a_k\right]\exp(-v(x))\,p_n(x) \tag{1.13}$$

where

$$R_{n,2}y = \frac{a_{n-1}}{A_{n-1}}\left[\frac{\mathrm{d}}{\mathrm{d}x} - B_n(x)\right]y \qquad n \geqslant 1 \tag{1.14}$$

and $A_0(x)/a_0$ in (1.13) is interpreted as

$$\frac{A_0(x)}{a_0} := \int_{-\infty}^{\infty} \exp(-v(y))\frac{v'(x)-v'(y)}{x-y}\,p_0^2(y)\,\mathrm{d}y \tag{1.15}$$

which amount to letting $n \to 0$ in $A_n(x)/a_n$.

Freud conjectured [17] that if $v(x) = |x|^\alpha$ then the recurrence coefficients have the limiting behaviour

$$a_n = \left[\frac{n\Gamma(1+\alpha/2)\Gamma(\alpha/2)}{\Gamma(\alpha+1)}\right]^{1/\alpha}[1+\mathrm{o}(1)] \tag{1.16}$$

for $\alpha > 0$. This conjecture has been proved for α an even integer in [13], for v a polynomial in [14], and for $\alpha > 0$ in [12]. The corresponding conjecture for the largest zero of $p_n(x)$ was proved independently by Rakhmanov [19] for $\alpha > 1$. For references to the extensive literature on this problem see [9–11, 24]. In section 4 we study properties of A_n and B_n when $v(x)$ is a polynomial of degree $2m, m = 1, 2, \ldots$. In section 5 we derive a Plancherel–Rotach asymptotic where the potential $v(x)$ is a polynomial of degree $2m$ from the differential equation obtained in this case. The asymptotics derived are for x around the largest zero where we show that the error term in (1.12) involves the smallest positive zero of the Airy function, $A(x)$. Recall that the solution to

$$Y''(x) + \frac{x}{3}Y(x) = 0 \tag{1.17}$$

which is bounded at $x = 0$ is a constant multiple of $A(x)$. We have already obtained preliminary results on the asymptotics in the rest of the complex plane, including the oscillatory range which will appear elsewhere. In section 6 we include an intuitive though not rigorous approach to the show how the Airy function appears. Our approach relies on the Coulomb fluid approximation [3, 4], which goes back to Dyson [6].

2. Differential relations and equations

In this section we discuss the raising and lowering operators and the second-order differential equation satisfied by the orthogonal polynomials.

Theorem 2.1. Let $w(x) = \exp(-v(x))$ and $v(x)$ be twice continuously differentiable convex functions on $(a, b) \subset (-\infty, \infty)$ and assume that w has moments of all orders. Then the polynomials $\{p_n(x)\}$ orthogonal with respect to $w(x)$ on (a, b) satisfy (1.5), where A_n and B_n are given by (1.6) and (1.7), provided that $w(a^+) = w(b^-) = 0$ and the integrals in (1.6) and (1.7) exist.

Proof. Since $p_n'(x)$ is a polynomial of degree $n - 1$, it can be expanded as

$$p_n'(x) = \sum_{k=0}^{n-1} c_{n,k}\, p_k(x). \tag{2.1}$$

Using the orthogonality relation (1.1) and integration by parts we see that

$$c_{n,k} = \int_{-\infty}^{\infty} p_n'(y) p_k(y) w(y)\, \mathrm{d}y = -\int_{-\infty}^{\infty} p_n(y)[p_k'(y) - p_k(y)v'(y)]w(y)\, \mathrm{d}y$$

hence the term involving p_k' vanishes. It follows that the right-hand side of (2.1) is

$$\int_{-\infty}^{\infty} p_n(y)\left[\sum_{k=0}^{n-1} p_k(x) p_k(y)\right] v'(y) w(y)\, \mathrm{d}y.$$

The above quantity vanishes if $v'(y)$ is replaced by $v'(x)$, hence the right-hand side of (2.1) is

$$\int_{-\infty}^{\infty} \left[\sum_{k=0}^{n-1} p_k(x) p_k(y)\right] [v'(y) - v'(x)] p_n(y) w(y)\, \mathrm{d}y.$$

Formula (1.5) now follows from the Christoffel–Darboux formula [22].

We next establish the useful formula

$$B_n(x) + B_{n+1}(x) = \frac{x - \alpha_n}{a_n} A_n(x) - v'(x). \tag{2.2}$$

To prove (2.2), use (1.7) and (1.3) to get

$$B_n(x) + B_{n+1}(x) = \int_{-\infty}^{\infty} p_n(y) \frac{v'(x) - v'(y)}{x - y} [y - \alpha_n] p_n(y) w(y)\, \mathrm{d}y$$
$$= \frac{x - \alpha_n}{a_n} A_n(x) + \int_{-\infty}^{\infty} [v'(y) - v'(x)] p_n^2(y) w(y)\, \mathrm{d}y.$$

The integral on the right-hand side of the above equation is

$$\int_{-\infty}^{\infty} v'(y) p_n^2(y) w(y)\, \mathrm{d}y - v'(x) = -v'(x) + 2\int_{-\infty}^{\infty} p_n(y)\, p_n'(y)\, w(y)\, \mathrm{d}y = -v'(x)$$

since the p_ns are orthonormal with respect to w. □

Remark. If w in theorem 2.1 does not vanish at the end points of the interval of orthogonality $[a, b]$ then A_n and B_n take the form

$$A_n(x) = \frac{a_n w(b^-) p_n^2(b^-)}{b - x} + \frac{a_n w(a^+)\, p_n^2(a^+)}{x - a} + a_n \int_a^b \frac{v'(x) - v'(y)}{x - y} p_n^2(y) w(y)\, \mathrm{d}y \qquad (2.3)$$

$$B_n(x) = \frac{a_n w(a^+) p_n(a^+) p_{n-1}(a^+)}{x - a} + \frac{a_n w(b^-) p_n(b^-) p_{n-1}(b^-)}{b - x}$$
$$+ a_n \int_a^b \frac{v'(x) - v'(y)}{x - y} p_n(y) p_{n-1}(y) w(y)\, \mathrm{d}y. \qquad (2.4)$$

This is the case for example with the Laguerre polynomials where $v(x) = x$, $a = 0$, $b = \infty$. In this case $p_n(x) = (-1)^n L_n(x)$, hence $A_n(x) = n/x$ and $B_n(x) = -n/x$.

Proof of (1.10). Eliminate $p_{n-1}(x)$ between (1.3) and (1.5) to get

$$\left(B_n(x) + \frac{\mathrm{d}}{\mathrm{d}x}\right) p_n(x) = \frac{A_n(x)}{a_n}[(x - \alpha_n) p_n(x) - a_{n+1} p_{n+1}(x)]. \qquad (2.5)$$

Use (2.2) to rewrite the above equation in the form (1.10). □

Theorem 2.2. Under the assumptions in theorem 2.1 the p_ns satisfy the factored equation

$$L_{2,n}\left(\frac{1}{A_n(x)}(L_{1,n} p_n(x))\right) = \frac{a_n}{a_{n-1}} A_{n-1}(x) p_n(x). \qquad (2.6)$$

Equivalently (2.6) is

$$p_n''(x) + P(x) p_n'(x) + Q(x) p_n(x) = 0 \qquad (2.7)$$

where

$$P(x) := -\left[v'(x) + \frac{A_n'(x)}{A_n(x)}\right] \qquad (2.8)$$

$$Q(x) := A_n(x)\left(\frac{B_n(x)}{A_n(x)}\right)' - B_n(x)[v'(x) + B_n(x)] + A_n(x) A_{n-1}(x) \frac{a_n}{a_{n-1}}$$
$$= B_n'(x) - B_n(x)\frac{A_n'(x)}{A_n(x)} - B_n(x)[v'(x) + B_n(x)] + \frac{a_n}{a_{n-1}} A_n(x) A_{n-1}(x). \qquad (2.9)$$

The Schrödinger form of (2.5) is

$$\Psi_n''(x) + V(x; n)\Psi_n(x) = 0 \qquad (2.10)$$

where

$$\Psi_n(x) := \frac{\exp[-v(x)/2]}{\sqrt{A_n(x)}} p_n(x) \tag{2.11}$$

and

$$V(x,n) = A_n(x)\left(\frac{B_n(x)}{A_n(x)}\right)' - B_n(x)[v'(x) + B_n(x)] + A_n(x)A_{n-1}(x)\frac{a_n}{a_{n-1}} + \frac{v''(x)}{2} \\ + \frac{1}{2}\left(\frac{A_n'(x)}{A_n(x)}\right)' - \frac{1}{4}\left[v'(x) + \frac{A_n'(x)}{A_n(x)}\right]^2. \tag{2.12}$$

Observe that $A_n(x) > 0$ due to the convexity of $v(x)$.

Theorem 2.3. If $v(x)$ is a polynomial of degree m then $A_n(x)$, $B_n(x)$, $A_n(x)P(x)$ and $A_n(x)Q(x)$ are polynomials of degrees $m-2$, $m-3$, $2m-3$, and $3m-6$, respectively.

Proof. Let $v'(x) = cx^{m-1} +$ lower order terms. Clearly (1.6) and (1.7) imply

$$A_n(x)/c = a_n x^{m-2} + \text{lower order terms}$$
$$B_n(x)/c = a_n^2 x^{m-3} + \text{lower order terms.}$$

It is now evident from (2.6) that the degree of $A_n(x)P(x)$ is $2m-3$. Furthermore

$$A_n(x)Q(x) := A_n(x)B_n'(x) - B_n(x)A_n'(x) - A_n(x)B_n(x)[v'(x) + B_n(x)] \\ + \frac{a_n}{a_{n-1}} A_n(x)A_{n-1}(x)$$

implies that $A_n(x)Q(x)$ is of degree $3m-6$. This completes the proof. □

Theorem 2.4. If $v(x)$ is a rational function, say

$$v'(x) = p(x) + r(x)/s(x) \tag{2.13}$$

where $p(x)$, $r(x)$ and $s(x)$ are polynomials of degrees p, r and s, respectively, then the orthogonal polynomials satisfy a linear second-order differential equation with polynomials coefficients.

Proof. It is clear that $p(x)$ in (2.12) contributes to $A_n(x)$ and $B_n(x)$ polynomials of degree $p-1$ and $p-2$, respectively. It is clear that if for a fixed positive integer k, the moments of $w(y)(y+c)^{-k}$ of all orders exist then

$$(x+c)^k \int_{-\infty}^{\infty} \frac{w(y)}{x-y} [(x+c)^{-k} - (y+c)^{-k}] p_n(y) p_{n-j}(y)\, \mathrm{d}y$$

is a polynomial of degree $k-1$ when $j=0$ and of degree at most $k-2$ if $j=1$. Thus $A_n(x)$ is the sum of polynomial of degree $p-1$ and a rational function while $B_n(x)$ is the sum of polynomial of degree $p-1$ and a rational function. □

3. Lie algebras generated by ladder operators

We first study the Lie algebra generated by $L_{n,1}$ and $L_{n,2}$ when $v'(x)$ and $B_n(x)$ are analytic functions in a domain containing the support of $w(x)$. Since

$$\exp(-v(x)/2)L_{n,1}(y\exp(v(x)/2)) = \left(\frac{\mathrm{d}}{\mathrm{d}x} + B_n(x) + \frac{1}{2}v'(x)\right)y$$
$$\exp(-v(x)/2)L_{n,2}(y\exp(v(x)/2)) = \left(-\frac{\mathrm{d}}{\mathrm{d}x} + B_n(x) + \frac{1}{2}v'(x)\right)y$$

then $\{L_{n,1}, L_{n,2}\}$ are equivalent to $\{M_{n,1}, M_{n,2}\}$,

$$M_{n,1} := \frac{\mathrm{d}}{\mathrm{d}x} \qquad M_{n,2}y := \left[B_n(x) + \frac{1}{2}v'(x)\right]y. \tag{3.1}$$

Set

$$f_1(x) := B_n(x) + \frac{1}{2}v'(x) \qquad f_{j+1}(x) = \frac{\mathrm{d}f_j(x)}{\mathrm{d}x} \qquad j > 0. \tag{3.2}$$

Let

$$M_{n,j}y = f_j(x)y \qquad j = 2, 3, \ldots. \tag{3.3}$$

It is easy to see that the Lie algebra generated by $\{M_{n,1}, M_{n,2}\}$ consists of $\{\mathrm{d}/\mathrm{d}x, f_j(x)\colon j = 1, 2, \ldots\}$. The M's satisfy the commutation relations

$$[M_{n,1}, M_{n,j}] = M_{n,j+1}, \; j > 1 \qquad [M_{n,j}, M_{n,k}] = 0 \qquad j, k > 1. \tag{3.4}$$

Theorem 3.1. The Lie algebra generated by $L_{n,1}$ and $L_{n,2}$ has dimension $2m+1$ when $v(x)$ is a polynomial of degree $2m$ for all n, $n > 0$.

Proof. Clearly the coefficient of x^{2m} in $v(x)$ must be positive and may be taken as one. Hence $B_n(x)$ is a polynomial of degree $2m-3$ with leading term $2ma_n^2x^{2m-3}$, so $f_1(x)$ has precise degree $2m-1$. Therefore $f_j(x)$ is a polynomial of degree $2m-j$, $j = 1, 2, \ldots, 2m$ and the theorem follows. □

The application of a theorem by Miller [16, ch 8], also stated as theorem 1 in [8], leads to the following result.

Theorem 3.2. Let f_1 be analytic in a domain containing $(-\infty, \infty)$. Then the Lie algebra generated by $M_{n,1}$ and $M_{n,2}$ is finite dimensional, say $k+2$, if and only if f_1 and its first k derivatives form a basis of solutions to

$$\sum_{j=0}^{k} a_j y^{(j)} = 0 \tag{3.5}$$

where $a_0, \ldots, a_k$ are constants which may depend on n, and $a_k \neq 0$.

This algebra has a very simple structure since all its elements, except for $\mathrm{d}/\mathrm{d}x$ commute. Any non-Abelian sub-algebras must contain $\mathrm{d}/\mathrm{d}x$.

4. Polynomials with exponential weights

In this section we investigate the asymptotics of $A_n(x)$ and $B_n(x)$ for large n.

Theorem 4.1. (Magnus [14].) Let $\{p_n(x)\}$ be orthonormal with respect to $\exp(-v(x))$ on $(-\infty,\infty)$ where $v(x)-x^{2m}$ is a polynomial of degree at most $2m-1$, $m=1,2,\ldots$. Then the recurrence coefficients satisfy

$$a_n = \left[\frac{m!(m-1)!}{(2m)!}n\right]^{1/(2m)}[1+\mathrm{o}(1)]. \tag{4.1}$$

Theorem 4.2. Let $\{p_n(x)\}$ be orthonormal with respect to an even weight function $w(x)$ and set

$$x^{2l}\, p_n(x) = \sum_{j=0}^{2l} c_{n,l,j}\, p_{n+2j-2l}(x). \tag{4.2}$$

Then $c_{n,l,j}$ is a homogeneous polynomial of degree $2l$ in $a_{n-2l+1}, a_{n-2l+2},\ldots,a_{n+2l}$ containing $\binom{2l}{j}$ non-zero terms, counting repetitions.

Proof. Use induction, (1.3) and (1.4).

It is clear that (4.2) implies

$$x^{2l+1}p_n(x) = \sum_{j=0}^{2l} c_{n,l,j}[a_{n+2j-2l+1}p_{n+2j-2l+1}(x) + a_{n+2j-2l}p_{n+2j-2l-1}(x)]. \tag{4.3}$$

□

Theorem 4.3. Let v be an even polynomial,

$$v(x) = \sum_{k=0}^{m} v_k x^{2k} \qquad v_m = 1. \tag{4.4}$$

Then as $n\to\infty$ we have

$$A_n(x) = 2a_n\sum_{k=0}^{m-1}\sum_{l=0}^{k}(k+1)v_{k+1}x^{2k-2l}a_n^{2l}\binom{2l}{l}[1+\mathrm{o}(1)] \tag{4.5}$$

$$B_n(x) = 2a_n^2\sum_{k=0}^{m-1}\sum_{l=0}^{k}(k+1)v_{k+1}x^{2k-2l-1}a_n^{2l}\binom{2l+1}{l}[1+\mathrm{o}(1)]. \tag{4.6}$$

Proof. It is clear from theorems 4.1 and 4.2 that as $n\to\infty$,

$$c_{n,j,l} = a_{2l}\binom{2l}{j}[1+\mathrm{o}(1)]. \tag{4.7}$$

Therefore

$$\begin{aligned} A_n(x) &= a_n\sum_{k=1}^{m}2kv_k\sum_{l=0}^{k-1}x^{2k-2l-2}\int_{-\infty}^{\infty}y^{2l}p_n^2(y)w(y)\,\mathrm{d}y \\ &= 2a_n\sum_{k=1}^{m}kv_k\sum_{l=0}^{k-1}x^{2k-2l-2}c_{n,l,l}[1+\mathrm{o}(1)]. \end{aligned}$$

The corresponding result for B_n follows similarly from (1.7) and (4.3). □

5. Plancherel–Rotach asymptotics

In this section we establish the Plancherel–Rotach asymptotics for Freud polynomials associated with the weights $\exp(-v(x))$ for even polynomials $v(x)$ of degree $2m$ with leading term x^{2m}. In our proof of the next result we shall use the Chu–Vandermonde sum [7]

$$ {}_2F_1(-n, a; c; 1) = \frac{(c-a)_n}{(c)_n}. \tag{5.1}$$

This is the terminating version of Gauss's theorem. A special case of (5.1) is

$$\sum_{l=0}^{m-1} \binom{2l}{l} 2^{-2l} = \sum_{l=0}^{m-1} \frac{(1/2)_l}{l!} = \lim_{\epsilon\to 0} {}_2F_1(1-m, 1/2; 1-m+\epsilon; 1) = \frac{(3/2)_{m-1}}{(m-1)!}. \tag{5.2}$$

Theorem 5.1. Let $v(x)$ be as in (4.4) and set

$$h_n(x) := A_n(x)\, A_{n-1}(x) - B_n^2(x) - B_n(x)\, v'(x) + \frac{A_n(x)}{2a_n} - \frac{v'^2(x)}{4} \tag{5.3}$$

$$X_n := \sqrt{4a_n^2 + 2a_n/A_n(2a_n)}. \tag{5.4}$$

Then

$$A_n(2a_n) = m(2a_n)^{2m-1}\, \frac{(3/2)_{m-1}}{(m-1)!}[1+\mathrm{o}(1)] = \frac{a_n^{2m-1}\,(2m)!}{(m-1)!^2}[1+\mathrm{o}(1)] \tag{5.5}$$

$$h_n(X_n) = \mathrm{O}(a_n^{-2}) + \mathrm{o}(A_n(X_n)). \tag{5.6}$$

Proof. From (4.5) it follows that

$$A_n(2a_n) = m(2a_n)^{2m-1} \sum_{l=0}^{m-1} \binom{2l}{l} 2^{-2l}$$

and (5.5) follows from (5.2) and (4.5). Next for sufficiently large n, $A_{n+1}(2a_n)/A_n(2a_n) \to 1$, so

$$h_n(x) = A_n^2(x) - [B_n + v'(x)/2]^2 + A_n(x)/(2a_n) + \mathrm{o}(A_n(x) + B_n(x))$$

for x sufficiently close to $2a_n$ and n large. Using (2.2) we get, for x as before,

$$h_n(x) = A_n^2(x)\left[\frac{1/2}{a_n A_n(x)} + 1 - \frac{x^2}{4a_n^2}\right] + \mathrm{o}(A_n(x) + B_n(x)). \tag{5.7}$$

Therefore

$$h_n(X_n)\,[1+\mathrm{o}(1)] = [A_n(X_n) - A_n(2a_n)]/(2a_n).$$

The above equation, the observation $B_n(x) = \mathrm{O}(A_n(x))$ and (4.5) imply (5.6) and the proof is complete. □

We now discuss the large n behaviour of $V(x; n)$. Recall that as $n \to \infty$, both a_n/a_{n-1} and $B_n(x)/B_{n-1}(x)$ tend to 1. Thus (2.2) implies

$$A_n(x)\left(\frac{B_n(x)}{A_n(x)}\right)'[1+\mathrm{o}(1)] = \frac{A_n(x)}{2a_n} - \frac{v''(x)}{2} + \frac{v'(x)}{2}\left[\frac{A_n'(x)}{A_n(x)}\right]$$

and

$$B_n(x)[v'(x) + B_n(x)][1+\mathrm{o}(1)] = \left(\frac{x}{2a_n} A_n(x)\right)^2 - \frac{[v'(x)]^2}{4}.$$

Therefore

$$V(x;n)[1+\mathrm{o}(1)] = A_n^2(x)\left[1-\frac{x^2}{4a_n^2}\right]+\frac{A_n(x)}{2a_n}+\frac{1}{2}\left(\frac{A_n'(x)}{A_n(x)}\right)'-\frac{1}{4}\left(\frac{A_n'(x)}{A_n(x)}\right)^2. \tag{5.8}$$

Every turning point of the differential equation (2.10) separates intervals of oscillation from non-oscillatory intervals. Since the zeros of the polynomials are real, simple and dense in $(-\infty,\infty)$ it then follows that the differential equation (2.10) has only two turning points at the largest and smallest zeros of $p_n(x)$. It is obvious from (5.8) that the dominant terms in $V(x;n)$ come from $h_n(x)$. Therefore the turning points are approximately $x=\pm X_n$, with X_n as defined in (5.4). We point out that the approximation of the largest zero of p_n by X_n of (5.4) is quite sharp. For example for the Hermite polynomials $v(x)=x^2$, $B_n(x)=0$ and $A_n(x)=2a_n$ and we find $X_n=\sqrt{2n+1}$, which is the precise approximation given in Szegö [22]. The procedure given in Nevai's survey article [17] for $v(x)=x^4$ and $v(x)=x^6$ is not as sharp. For example in the case of Hermite polynomials it gives the approximation $\sqrt{2n}$ for largest zeros of $H_n(x)$, as was pointed out by Nevai [17, (4.21.59)].

Theorem 5.2. Let the zeros of $p_n(x)$ be

$$x_{n,1} > x_{n,2} > \cdots. \tag{5.9}$$

Then

$$x_{n,k} = X_n - i_k\left(\frac{2a_n}{6A_n^2(2a_n)}\right)^{1/3} \tag{5.10}$$

where i_k is the kth positive zero of the Airy function [22].

This proves a conjecture in [5] that

$$x_{n,1} = X_n - i_1\left(\frac{2a_n}{6A_n^2(2a_n)}\right)^{1/3}. \tag{5.11}$$

Note that $A_n(2a_n)$ is given by (5.5). In the case of Hermite polynomials $a_n=\sqrt{n/2}$, $A_n(x)=2a_n$ and (5.10) reduces to the well known asymptotics in [22].

Proof of theorem 5.2. In the differential equation

$$y''+V(x;n)y=0 \tag{5.12}$$

let $x=X_n-\xi$. Clearly $B_n(x)=\mathrm{O}(A_n(x))$ and

$$\begin{aligned} h_n(x) &= A_n^2(x)\left[\frac{X_n^2-x^2}{4a_n^2}\right]+\frac{A(x)-A(X_n)}{2a_n}+\mathrm{o}(A_n(x)) \\ &= (X_n-x)\left[A_n^2(x)\frac{X_n+x}{4a_n^2}+\frac{A(x)-A(X_n)}{2a_n(X_n-x)}\right]+\mathrm{o}(A_n(x)) \end{aligned} \tag{5.13}$$

hold when n is large but ξ lies in a compact set. On the other hand (5.8) implies $V(x;n)=h_n(x)+\mathrm{O}(x^{-2})$ as $x\to\infty$. Therefore (5.12) becomes

$$\frac{\mathrm{d}^2y}{\mathrm{d}\xi^2}+y[\xi A_n^2(2a_n)/a_n]=[\mathrm{o}(A_n(2a_n))]y.$$

After we replace ξ by $\zeta(3a_n/A_n^2(2a_n)$ we transform the above differential equation to

$$\frac{\mathrm{d}^2y}{\mathrm{d}\zeta^2}+(\zeta/3)y=[\mathrm{o}(a_n/A_n(2a_n))]y=[\mathrm{o}(a_n^{2-2m})]y. \tag{5.14}$$

The next step is to apply perturbation theory to (5.14) using the Airy equation (1.17) as the model equation. After straightforward calculations we establish (5.10) for $k=1$ and the general case will appear elsewhere. □

6. Asymptotics near the edges

This section contains a plausible and physical explanation for the appearance of the Airy function and its zeros in the asymptotics of polynomials orthogonal with respect to exponential weights.

Recall that we assumed that $v(x)$ is twice differentiable and convex. Furthermore, we shall require that the associated moment problem is determinate. It appears that things are not quite clear in the indeterminate cases. For terminology and the literature on the moment problem we refer the interested reader to the books by Akhiezer [1] and Shohat and Tamarkin [21]. First we expect that $B_n(x)$ for sufficiently large n is a slowly varying function of n and $B_n(x) \approx B_{n+1}(x)$. Clearly a rigorous estimate, uniform in x, of $|B_{n+1}(x) - B_n(x)|$ is highly desirable. Second, $A_n(x)$ is also expected to be a slowly varying function of n and $A_n(x) \approx A_{n-1}(x)$. However, note that as the polynomials are rapidly varying functions of n, we do not have the approximation, $p_n(x) \approx p_{n\pm1}(x)$. Third, $a_n \approx a_{n-1}$ which is clear from [12]. Now using these observations we find

$$B_n(x) \approx \frac{x-\alpha_n}{2\sqrt{\alpha_n}} A_n(x) - \frac{v'}{2}. \tag{6.1}$$

Now a simple computation using the above approximation shows,

$$A_n(x)\left(\frac{B_n(x)}{A_n(x)}\right)' \approx \frac{A_n(x)}{2\sqrt{\beta_n}} - \frac{v''(x)}{2} + \frac{v'(x)}{2}\left[\frac{A_n'(x)}{A_n(x)}\right]. \tag{6.2}$$

Also

$$B_n(x)(v'(x) + B_n(x)) \approx \left(\frac{x-\alpha_n}{2\sqrt{\beta_n}} A_n(x)\right)^2 - \frac{[v'(x)]^2}{4}. \tag{6.3}$$

Finally, putting all the pieces together we see from (2.12) that

$$V(x;n) \approx A_n^2(x)\left[1 - \left(\frac{x-\alpha_n}{2\sqrt{\beta_n}}\right)^2\right] + \frac{A_n(x)}{2\sqrt{\beta_n}} + \frac{1}{2}\left(\frac{A_n'(x)}{A_n(x)}\right)' - \frac{1}{4}\left(\frac{A_n'(x)}{A_n(x)}\right)^2. \tag{6.4}$$

Now we need to relate $A_n(x)$ to the Coulomb fluid density, $\sigma(x)$. It was shown in [4] that

$$A_n(x) \approx 2\pi\sqrt{\beta_n}\frac{\sigma(x)}{\sqrt{(b-x)(x-a)}} = 2\pi a_n \frac{\sigma(x)}{\sqrt{(b-x)(x-a)}}. \tag{6.5}$$

From these we find,

$$V(x;n) \approx \frac{\pi\sigma(x)}{\sqrt{(b-x)(x-a)}} + \pi^2\sigma^2(x) + \frac{1}{2}\left(\frac{A_n'(x)}{A_n(x)}\right)' - \frac{1}{4}\left(\frac{A_n'(x)}{A_n(x)}\right)^2. \tag{6.6}$$

Now under the assumptions on $v(x)$, we find in the Coulomb fluid approximation

$$\sigma(x) \approx G(a,b)\sqrt{b-x} \qquad \text{as } x \to b \tag{6.7}$$

where $G(a,b)$ has an integral representation and can be shown to positive due to the convexity of $v(x)$. See [4] regarding the integral representation and positivity of $G(a,b)$. Thus

$$V(x;n) \approx \frac{\pi G(a,b)}{\sqrt{b-a}} + \pi^2 G^2(a,b)(b-x) \tag{6.8}$$

as $x \to b$ and the differential equation (2.10) becomes

$$\Psi_n''(x) + \left[\frac{\pi G(a,b)}{\sqrt{b-a}} + \pi^2 G^2(a,b)(b-x)\right]\Psi_n(x) = \mathrm{o}(\Psi_n(x)). \tag{6.9}$$

At this stage we make the change of variable

$$t := \left(\frac{3/\pi}{G(a,b)}\right)^{1/3}\left[\frac{1}{\sqrt{b-a}} + \pi G(a,b)(b-x)\right] \tag{6.10}$$

then

$$\frac{\mathrm{d}^2\Psi_n(t)}{\mathrm{d}t^2} + \frac{t}{3}\Psi_n(t) = \mathrm{o}(\Psi_n(t)). \tag{6.11}$$

Therefore one would expect Ψ_n as a function of t to have the property

$$\lim_{n\to\infty}\Psi_n(t)/C_n = A(t) \tag{6.12}$$

for some numerical sequence C_n and the limit holds uniformly on compact subsets of the complex t-plane. The turning point of (6.9) near $x = b$ is

$$X_n(b) := b + \frac{(b-a)^{-1/2}}{\pi G(a,b)}. \tag{6.13}$$

This suggests that the zeros of $p_n(x)$, arranged as in theorem 5.2, have the limiting behaviour

$$x_{n,k} = X_n(b) - \frac{i_k}{3^{1/3}}(\pi\, G(a,b))^{-2/3}[1+\mathrm{o}(1)]. \tag{6.14}$$

As an example consider the Freud weight $w(x) = \exp(-|x|^\alpha)$. Here $v(x) = |x|^\alpha$ and $a = -b$. For $\alpha \geqslant 2$, v is convex. It is known that in this case

$$\pi G(-b,b) = \frac{2^{+\alpha-3/2}\Gamma(\alpha+1)}{\Gamma^2(\alpha/2)}b^{\alpha-3/2} \tag{6.15}$$

see for example [3]. Furthermore

$$b^\alpha = \frac{\Gamma^2(\alpha/2)\,2^{\alpha-1}n}{\Gamma(\alpha)}[1+\mathrm{o}(1)] \tag{6.16}$$

[12, 19]. In view of (6.13) we take

$$X_n(b) = b + \frac{\Gamma^2(\alpha/2)}{\sqrt{2b}\Gamma(\alpha+1)}\left[\frac{2}{b}\right]^{\alpha-3/2}. \tag{6.17}$$

Now (6.14) becomes

$$X_n(b) = b\left[1 + \frac{1}{2\alpha n}\right]. \tag{6.18}$$

This suggests

$$x_{n,k} = b\left[1+\frac{1}{2\alpha n}\right] - \frac{i_k}{3^{1/3}}\left(\frac{2}{\alpha}\right)^{2/3}\left(\frac{\Gamma^2(\alpha/2)}{2\Gamma(\alpha)}\right)^{1/\alpha} n^{1/\alpha-\frac{2}{3}}[1+\epsilon_n] \tag{6.19}$$

where b is given by (6.16) and $\epsilon_n \to 0$ as $n \to \infty$. We believe ϵ_n to be positive. In the case of Hermite polynomials $\alpha = 2$ and (6.18) reduces to what one gets from theorem 8.22.9 in Szegö [22]. Szegö shows that $\epsilon_n > 0$ for the Hermite polynomials.

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Note added in proof. It has come to our attention that the differential equations established here have been established for polynomial v by Bonan and Clark (Bonan S S and Clark D S 1990 *J. Approx. Theory* **63** 210) and by Bauldrey (Bauldrey N C 1990 *J. Approx. Theory* **63** 225) where they were used to establish bounds from Freud polynomials. Neither Bauldrey nor Bonan and Clark identified the creation operators, Lie algebras, nor did they study the connection with the zero of the Airy function.

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